Nonlinear Ginzburg-Landau-type approach to quantum dissipation

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We formally derive two nonlinear Ginzburg-Landau type models starting from the Wigner-Fokker-Planck system, which rules the evolution of a quantum electron gas interacting with a heat bath in thermodynamic equilibrium. These models mainly consist of a quantum, dissipative $O(\hbar^3)$ hydrodynamic/$O(\hbar^4)$ stochastic correction to the frictional (Caldeira-Leggett-)Schrödinger equation. The main ingredient lies in the use of the hydrodynamic/stochastic fluid model approach associated with the quantum Fokker-Planck equation and the identification of the associated pressure field. Then, Madelung transformations set the problem in the Schrödinger picture of dissipative quantum mechanics. We also describe the stationary dynamics associated with both systems.

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I. INTRODUCTION AND SETTING OF THE PROBLEM

The mathematical modeling and analysis of quantum dissipation phenomena have experienced a great impulse in past years. The inclusion of dissipation within quantum mechanics is mainly based on a system-plus-reservoir formulation, which means that energy is lost by the system and absorbed by the environment (for instance, a semiconductor device in which doped regions are considered as electron reservoirs injecting electrons into the active regions). Then, the physical scenario of which these phenomena take part is that of open quantum systems, i.e., a particle ensemble interacting dissipatively with an idealized heat bath of harmonic oscillators. The effect of the thermal bath on the motion of the particles is typically described by two parameters: the bath temperature and the friction constant. In this direction, the qualitative study of different approaches to the quantum Fokker-Planck master equation in the Wigner representation (or Kramers equation) has been the subject of several recent works, e.g., Refs. [1–11]. In this paper we assume that the interaction of an infinite in extent 1D (for notational convenience) quantum gas of spinless fermions with a thermal bath (subject to moderate/high temperatures) is described by the following Wigner-Fokker-Planck equation with nonvanishing friction mechanism (see Ref. [4] for a systematic derivation)

$$W_t + \xi W_x + \Theta_\hbar[V]W = L_{QFP}[W] \qquad (1)$$

with

$$L_{QFP}[W] = \frac{D_{pp}}{m^2} W_{\xi\xi} + 2\lambda(\xi W)_\xi + \frac{2}{m} D_{pq} W_{\xi x} + D_{qq} W_{xx}, \qquad (2)$$

where $W = W(x,\xi,t)$ is the (quasi)probability distribution function, x and ξ hold for the coordinates of the electron gas and of the bath, respectively,

$$D_{pp} = \eta k_B T, \quad D_{pq} = \frac{\eta \Omega \hbar^2}{12\pi m k_B T}, \quad D_{qq} = \frac{\eta \hbar^2}{12 m^2 k_B T} \qquad (3)$$

are phenomenological constants related to the interactions,

$$\lambda = \frac{\eta}{2m} \qquad (4)$$

is the friction coefficient, m is the effective mass of the electrons, η is the damping/coupling constant of the bath, Ω is the cut-off frequency of the reservoir oscillators, k_B is the Boltzmann constant, T is the bath temperature, and where

$$(\Theta_\hbar[V]W)(x,\xi,t) = \frac{i}{2\pi\hbar} \int_{\mathbb{R}^2} [V(x_+,t) - V(x_-,t)]$$
$$\times W(x,\xi',t) e^{-i(\xi-\xi')y} d\xi' dy \qquad (5)$$

is a pseudo-differential operator associated with the given potential V. This operator can make the equation to become nonlinear in virtue of the chosen potential. Here, $\hbar$ denotes the reduced Planck constant and x_+ and x_- are the shifted position variables

$$x_+ = x + \frac{\hbar}{2m} y, \quad x_- = x - \frac{\hbar}{2m} y. \qquad (6)$$

This or similar simplified models are at the basis of quantum kinetics of open systems, microelectronics and nanosolid physics. In fact, they are being currently explored in various fields of scientific interest mainly stemming from mesoscopic mechanics, such as quantum Brownian motion, quantum optics, semiconductor device applications, quantum measurement theory, decoherence and emergent classicality, beam propagation in accelerators or activated chemical processes, among others.

When D_{pq} and D_{qq} are set to zero, the well-known Caldeira-Leggett master equation [12] is obtained. It is remarkable the fact that the Wigner-Fokker-Planck equation (1) and (2) is charge-preserving, i.e.,

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$$q = \int_{\mathbb{R}} n(x,t)\,dx$$

is an invariant of motion, where

$$n(x,t) = \int_{\mathbb{R}} W(x,\xi,t)\,d\xi \tag{7}$$

stands for the electron position density. However, the diffusive character of the system prevents ξ and ξ^2 to be collisional invariants. Indeed, defining the current and the electron kinetic energy densities by

$$J(x,t) = \int_{\mathbb{R}} \xi W(x,\xi,t)\,d\xi, \tag{8}$$

$$E(x,t) = \frac{1}{2}\int_{\mathbb{R}} \xi^2 W(x,\xi,t)\,d\xi, \tag{9}$$

respectively, one easily finds

$$\int_{\mathbb{R}} \xi L_{QFP}[W]\,d\xi = -2\lambda J - \frac{2}{m}D_{pq}n_x + D_{qq}J_{xx}, \tag{10}$$

$$\frac{1}{2}\int_{\mathbb{R}} \xi^2 L_{QFP}[W]\,d\xi = \frac{D_{pp}}{m^2}n - 4\lambda E - \frac{2}{m}D_{pq}J_x + D_{qq}E_{xx}. \tag{11}$$

Thus, the momentum-and-energy conservation kinetic identities do not hold in the frictional/diffusive case.

Another interesting aspect is that the Wigner-Fokker-Planck equation (1) and (2) is written in so-called Lindblad form (see Ref. [13], Sec. 2 of Ref. [1]) provided that the following constraint

$$D_{pp}D_{qq} \geq D_{pq}^2 - \tfrac{1}{4}\hbar^2\lambda^2$$

is satisfied by the diffusion coefficients. Equivalently, in terms of the original thermal bath constants this condition reads

$$\hbar\Omega \leq \sqrt{3}\,\pi k_B T$$

for nontrivial situations ($\eta \neq 0$). Both relations can be found in [5,14]. Then, the associated density matrix operator (a linear, nonnegative, self-adjoint trace class operator) $R(t):L^2(\mathbb{R}) \to L^2(\mathbb{R})$, defined by

$$(R(t)f)(x) = \int_{\mathbb{R}_y} f(y)\rho(x,y,t)\,dy,$$

preserves positivity under temporal evolution:

$$\frac{\partial\rho}{\partial t} = -\frac{i}{\hbar}(H_x - H_y)\rho - \lambda(x-y)\cdot(\nabla_x - \nabla_y)\rho$$

$$+ \left(D_{qq}|\nabla_x + \nabla_y|^2 - \frac{D_{pp}}{\hbar^2}|x-y|^2 \right.$$

$$\left. + \frac{2i}{\hbar}D_{pq}(x-y)\cdot(\nabla_x + \nabla_y) \right)\rho.$$

Here, $\rho = \rho(x,y,t) \in L^2(\mathbb{R}_x \times \mathbb{R}_y)$ is the density matrix function and

$$H = -\frac{\hbar^2}{2m}\Delta_x + V(x,t)$$

is the electron Hamiltonian, H_x and H_y standing for copies of H acting on the x and y variables, respectively. Accordingly, the problem is physically meaningful and mathematically consistent (see [1,2] for details).

Associated with the quantum Fokker-Planck system (1)–(6) there is the following macroscopic fluid model:

$$n_t + (nu)_x = D_{qq}n_{xx}, \tag{12}$$

$$u_t + \left(u - 2D_{qq}\frac{n_x}{n} \right)u_x = -\frac{1}{m}V_x - \frac{1}{n}(2E - nu^2)_x - 2\lambda u$$

$$-\frac{2}{m}D_{pq}\frac{n_x}{n} + D_{qq}u_{xx}, \tag{13}$$

which yields the propagation laws for the density $n(x,t)$ [cf. Eq. (7)] and the fluid mean velocity

$$u(x,t) = \frac{J(x,t)}{n(x,t)} \tag{14}$$

and incorporates quantum dissipation within the context of Fokker-Planck scattering. This model is to be compared to the hydrodynamic formulation of pure state de Broglie–Bohm quantum mechanics [15,16], consisting of the following current continuity equation and momentum equation (see also Refs. [17–19] for recent analysis):

$$n_t + (nu)_x = 0,$$

$$u_t + uu_x = -\frac{1}{m}(V+Q)_x,$$

where Q is the quantum potential of Bohm defined by

$$Q = -\frac{\hbar^2}{2m}\frac{(\sqrt{n})_{xx}}{\sqrt{n}}. \tag{15}$$

Notice that the balance equation (12) for $n(x,t)$ is allowed to depend not only on the gas density, as occurs in the usual kinetic theory of gases and liquids, but also on its gradient. In fact, density-gradient theories have already proved to have a significant range of physical applicability. The drift-diffusion system (12) and (13) has been recently dealt with in

[6,7], where existence of classical solutions to the stationary problem as well as exponential convergence of the solutions towards the thermal equilibrium state in the long time were shown.

Far from the standard arguments connecting both quantum-mechanical Wigner and Schrödinger pictures, based on density matrix approaches, the hydrodynamic equations (12) and (13) become now the key tool in our derivations (in the spirit of Refs. [20–22]), as we shall see later on. Actually we show that, for a large class of quantum mixed states, the Wigner-Fokker-Planck system can be reduced to an exact, effective nonlinear Ginzburg-Landau-type equation which accommodates quantum dissipation via the introduction of diffusion currents. In the hydrodynamic approach, this equation is even shown to contain additional nonlinearities coming out from an adequate identification of the pressure field. The connection between open quantum systems and Ginzburg-Landau theories has been already discussed in different physical contexts, e.g., trapped modes of cold, dilute, weakly interacting Bose gases [23], evaporate cooling in Bose-Einstein condensates [24], or cosmological quantum kinetic theory [25].

The paper is structured as follows: In Sec. II we give a hydrodynamic approach to quantum dissipation in the Schrödinger framework. To this aim, we first introduce the quantum fluid model associated with the one-dimensional (1D) Wigner-Fokker-Planck system in ξ-moments picture. Then, we proceed to the identification of the classical and quantum fluid dynamical pressures. Also, we justify the presence of the quantum Bohm potential in the fluid equations as well as that of an additional effective pressure potential. Finally, the (formal) nonperturbative derivation of an effective, dissipative nonlinear Ginzburg-Landau-type equation for the quantum gas is carried out. Section III is devoted to investigate a different (stochastic) approach leading to a nonlinear, dissipative Schrödinger-type equation. To the best of our knowledge, both the hydrodynamic and stochastic Ginzburg-Landau type models derived in this paper have not been dealt with before in the literature. In Sec. IV we make a detailed analysis of the existence of stationary solutions to the nonlinear Ginzburg-Landau-type equations derived in Sec. II and III. Also, for some particular cases we find explicit solutions and describe their main dynamical properties. Finally, some technical remarks on the 3D derivations are collected in the Appendix.

II. THE HYDRODYNAMIC APPROACH

In this section we are concerned with a hydrodynamic (quantum fluid) approach to the Wigner-Fokker-Planck system. Actually, we derive a nonlinear Ginzburg-Landau-type equation [see Eq. (46) below] which accounts for quantum friction and diffusion effects. Besides the well-known logarithmic nonlinearity first studied by Byalinicki-Birula and Mycielski in Ref. [26] and the nonlinear, frictional term ηS proposed by Kostin in [27], where S stands for the phase of the wave function, our equation retains a $\mathcal{O}(\hbar^3)$ nonlinear complex potential describing quantum position diffusion. To proceed, we first recompute the fluid equations associated

with the quantum Fokker-Planck system in terms of the current velocity and the classical and quantum parts of the pressure field [cf. Eq. (31) below]. Then, we use the polar form of the wave function to transform the fluid equations into a Bohm-Madelung-type Schrödinger equation which includes dissipative and hydrodynamic terms.

A. The fluid model

Following the standard ξ-moments picture method [cf. Eqs. (7)–(11)], one gets from the Wigner-Fokker-Planck equation (1)–(2) the following quantum hydrodynamic system of irrotational flow equations

$$n_t + (nu)_x = D_{qq} n_{xx}, \tag{16}$$

$$J_t + 2E_x + \frac{1}{m} nV_x = -2\lambda J - \frac{2}{m} D_{pq} n_x + D_{qq} J_{xx}. \tag{17}$$

Equation (16) is easily recognized as a Fokker-Planck equation. By using Eqs. (14) and (16), the current equation (17) may be recast in terms of the fluid mean velocity as

$$u_t + uu_x = -\frac{1}{m} V_x - \frac{1}{n}(P_u)_x - 2\lambda u - \frac{2}{m} D_{pq} \frac{n_x}{n} + F(n,u), \tag{18}$$

where we have identified the scalar pressure field as

$$P_u(x,t) = 2E(x,t) - n(x,t)u(x,t)^2 \tag{19}$$

as can be simply checked after comparison with Eq. (13) (see Ref. [28] for a general setting). Here, D_{qq} plays the role of a kinematic viscosity and along with D_{pq} measures in some sense the strength of gradient effects in the gas-reservoir system. On the other hand, the cross diffusion (D_{pq} term) gives rise to so-called Drude correction, which takes into account temperature effects. This term vanishes in the high-temperature limit $k_B T \gg \hbar \Omega$. Finally, $F(n,u)$ represents the dissipative force given by

$$F(n,u) = D_{qq} \left(2\frac{n_x}{n} u_x + u_{xx} \right). \tag{20}$$

Under this hydrodynamic approach the quantum fluid can be seen as the medium in which the particles are transported. Actually, the electron ensemble assumes the form of a highly localized inhomogeneity moving with the local fluid mean velocity.

Define the current velocity

$$v := u - D_{qq} \frac{n_x}{n}. \tag{21}$$

Then, Eq. (16) becomes the usual continuity equation of fluid mechanics (mass conservation law)

$$n_t + (nv)_x = 0. \tag{22}$$

Also, $v(x,t)$ satisfies the following partial differential equation

$$v_t + vv_x = -\frac{1}{m}V_x - \frac{1}{n}(P_v)_x - 2\lambda v - \frac{2}{m}D_{pq}\frac{n_x}{n} + G(n,v),$$

(23)

where the position-diffusion kernel $G(n,v)$ is now given by

$$G(n,v) = 2D_{qq}\left[v_{xx} + 2\frac{n_x}{n}v_x + \frac{n_{xx}}{n}v - \lambda\frac{n_x}{n}\right] + D_{qq}^2\frac{n_{xxx}}{n}.$$

(24)

Note that we have used Eqs. (18), (21), and (22).

B. Hydrodynamic to quantum transition: A quantum potential and diffusion representation

Now we are concerned with the derivation of an extended Bohm's causal approach to the viscous quantum hydrodynamic system introduced above. To proceed, we first observe that the moment system (16)–(20) [alternatively Eqs. (22)–(24)] is not closed, as the expression for the pressure involves the second order moment $\int_{\mathbb{R}_\xi}\xi^2 W(x,\xi,t)d\xi$. Then, some "admissible" closure relations are required.

Denote $\mathrm{Re}(\varphi)$ and $\mathrm{Im}(\varphi)$ the real and imaginary parts of the complex function φ, respectively. We also denote $\bar\varphi$ the complex conjugate of φ. Following the ideas in Ref. [21], we now suppose that $W(x,\xi,t)$ is the Wigner distribution associated with a quantum mixture of (complex) states $\psi_n(x,t)$, that is,

$$W(x,\xi,t) = \frac{1}{2\pi}\sum_{k\geqslant 1}\lambda_k\int_{\mathbb{R}}\bar\psi_k(x_-,t)\psi_k(x_+,t)e^{-iy\xi}dy$$

with the notation introduced in Eq. (6), where the λ_k's are the occupation probabilities

$$\lambda_k \geqslant 0, \quad \sum_{k\geqslant 1}\lambda_k = 1.$$

Also, we consider the local diffusion current $j(x,t)$ to be defined by

$$j := nv = J - D_{qq}n_x$$

$$= \frac{\hbar}{m}\sum_{k\geqslant 1}\lambda_k\mathrm{Im}(\bar\psi_k(\psi_k)_x) - D_{qq}\sum_{k\geqslant 1}\lambda_k(|\psi_k|^2)_x, \quad (25)$$

so as to fit Fick's law. Then, the pressure field (19) can be rewritten in terms of the quantum states ψ_k (omitting the (x,t) dependence for the sake of simplicity) as

$$P_v = \int_{\mathbb{R}}\xi^2 W d\xi - \frac{j^2}{n}$$

$$= -\frac{\hbar^2}{2m^2}\sum_{k\geqslant 1}\lambda_k\{\mathrm{Re}[\bar\psi_k(\psi_k)_{xx}] - |(\psi_k)_x|^2\}$$

$$- \frac{1}{\sum_{k\geqslant 1}\lambda_k|\psi_k|^2}\left(\frac{\hbar^2}{m^2}\left\{\sum_{k\geqslant 1}\lambda_k\mathrm{Im}[\bar\psi_k(\psi_k)_x]\right\}\right)^2$$

$$- \frac{4\hbar}{m}D_{qq}\left\{\sum_{k\geqslant 1}\lambda_k\mathrm{Im}[\bar\psi_k(\psi_k)_x]\right\}\left\{\sum_{k\geqslant 1}\lambda_k\mathrm{Re}[\bar\psi_k(\psi_k)_x]\right\}$$

$$+ 4D_{qq}^2\left\{\sum_{k\geqslant 1}\lambda_k\mathrm{Re}[\bar\psi_k(\psi_k)_x]\right\}^2\Bigg).$$

By writing the quantum states in polar form

$$\psi_k(x,t) = A_k(x,t)e^{i/\alpha S_k(x,t)},$$

where $A_k(x,t)$ and $S_k(x,t)$ hold for the amplitude (modulus) and the phase (argument) of $\psi_k(x,t)$, respectively, and where

$$\alpha = 2mD_{qq},$$

(26)

we can split

$$P_v = P_v^c + P_v^q,$$

where

$$P_v^c = \frac{\hbar^2}{m^2\alpha^2}\left\{\sum_{k\geqslant 1}\lambda_k A_k^2(S_k)_x^2 - \frac{\left[\sum_{k\geqslant 1}\lambda_k A_k^2(S_k)_x\right]^2}{\sum_{k\geqslant 1}\lambda_k A_k^2}\right\}$$

is the classical part of the pressure and

$$P_v^q = \frac{\hbar^2}{2m^2}\sum_{k\geqslant 1}\lambda_k[(A_k)_x^2 - A_k(A_k)_{xx}]$$

$$+ \frac{2\hbar}{m^2}\frac{\left[\sum_{k\geqslant 1}\lambda_k A_k^2(S_k)_x\right]\left[\sum_{k\geqslant 1}\lambda_k A_k(A_k)_x\right]}{\sum_{k\geqslant 1}\lambda_k A_k^2}$$

$$- 4D_{qq}^2\frac{\left[\sum_{k\geqslant 1}\lambda_k A_k(A_k)_x\right]^2}{\sum_{k\geqslant 1}\lambda_k A_k^2}$$

(27)

is the quantum correction of order $\hbar^4$ [recall that $D_{qq} = O(\hbar^2)$] to the classical pressure (see Refs. [17,29] for a physical justification in a nondissipative context).

Some observations are now in order:

(i) In the absence of dissipation in the x direction ($D_{qq} = 0$), the difference between the classical zero-temperature Euler equations and the hydrodynamic equations (16)–(20) [or Eqs. (22)–(24)] mainly lies in the quantum contribution

$$\frac{\hbar^2}{2m^2}\sum_{k\geqslant 1}\lambda_k[(A_k)_x^2 - A_k(A_k)_{xx}]$$

of order $\hbar^2$ appearing in the velocity equation, which plays a central role in the quantum potential approach.

(ii) The classical pressure P_v^c obviously vanishes in the purely quantum (single-state) case.

(iii) In standard dissipative quantum mechanics, the diffusion coefficient in the Fokker-Planck equation (16) reads

$$D = \frac{\hbar}{2m},$$

which yields the usual value $\alpha = \hbar$. We are, however, concerned with a generalized, nonstandard quantum setting (26) (see Ref. [30] for details) involving D_{qq} as the diffusion coefficient of the process.

Now we consider the average velocities associated with the wave functions ψ_k, defined by

$$v_k = \frac{1}{m}(S_k)_x.$$

It is a simple matter to corroborate that P_v^c can be rewritten in terms of the velocity variance as

$$P_v^c = \frac{\hbar^2}{\alpha^2} n(\langle v^2 \rangle - \langle v \rangle^2),$$

where the values $\langle v^\gamma \rangle$ represent the statistical averages given by

$$\langle v^\gamma \rangle = \frac{1}{n} \sum_{k \geq 1} \lambda_k A_k^2 v_k^\gamma.$$

In this way, we recover the standard expression for the classical pressure function (see Ref. [28]). Also, Eq. (27) now reads

$$P_v^q = \frac{\hbar^2}{2m^2} \sum_{k \geq 1} \lambda_k [(A_k)_x^2 - A_k(A_k)_{xx}] + \frac{2\hbar}{m} \left[\sum_{k \geq 1} \lambda_k A_k(A_k)_x \right]$$

$$\times \langle v \rangle - 4D_{qq}^2 \frac{\left[\sum_{k \geq 1} \lambda_k A_k(A_k)_x \right]^2}{\sum_{k \geq 1} \lambda_k A_k^2}. \tag{28}$$

In order to close the moment system we need some continuity equations relating the classical and quantum pressures to the particle density. To this aim, we first make the standard assumption that the classical pressure only depends on the position density and then consider that the amplitudes are all equal, i.e.,

$$P_v^c = P_v^c(n), \tag{29a}$$

$$A_k(x,t) = A(x,t) = \sqrt{n(x,t)} \quad \forall k \in \mathbb{N}, \tag{29b}$$

so that Eq. (28) becomes

$$P_v^q = \frac{\hbar^2}{2m^2} [(\sqrt{n})_x^2 - \sqrt{n}(\sqrt{n})_{xx}]$$

$$+ \frac{2\hbar}{m} \sqrt{n}(\sqrt{n})_x \langle v \rangle - 4D_{qq}^2(\sqrt{n})_x^2$$

$$= -\frac{\hbar^2}{4m^2} n \left(\frac{n_x}{n} \right)_x + \frac{\hbar}{m} n_x \langle v \rangle - D_{qq}^2 \frac{n_x^2}{n}. \tag{30}$$

In the P_v^c-P_v^q representation, the current velocity equation (23) now reads

$$v_t + v v_x = -\frac{1}{m} V_x - \frac{1}{n}(P_v^c)_x - \frac{1}{n}(P_v^q)_x - 2\lambda v$$

$$- \frac{2}{m} D_{pq} \frac{n_x}{n} + G(n,v)$$

$$= -\frac{1}{m} V_x - \frac{1}{n}(P_v^c)_x - \frac{1}{n}(P_v^q)_x$$

$$- 2\left(\frac{D_{pq}}{m} + \lambda D_{qq} \right) \frac{n_x}{n} + \frac{2}{n}(D_{qq} j_{xx} - \lambda j)$$

$$+ D_{qq}^2 \left(\frac{n_{xxx}}{n} \right) \tag{31}$$

after straightforward calculations, where we have used the first identity in Eq. (25).

Now, taking into account Eqs. (30) and (31), the physical assumptions (29a) and (29b) and the following identities involving the average velocity $\langle v \rangle$:

$$\frac{1}{n}(P_v^q)_x = \left(\frac{\hbar^2}{4m^2} - D_{qq}^2 \right) \frac{n_x}{n^2} \left[2n_{xx} - \frac{n_x^2}{n} \right] - \frac{\hbar^2}{4m^2} \left(\frac{n_{xxx}}{n} \right)$$

$$+ \frac{\hbar}{m} \frac{1}{n}(n_x \langle v \rangle)_x,$$

$$j = \frac{\hbar}{\alpha} n \langle v \rangle - D_{qq} n_x,$$

$$v_t = \frac{\hbar}{\alpha} \langle v \rangle_t + \frac{\hbar}{\alpha} D_{qq} \left[\left(\frac{n_x}{n} \langle v \rangle \right)_x + \langle v \rangle_{xx} \right] - D_{qq}^2 \left(\frac{n_{xx}}{n} \right)_x,$$

$$v v_x = \frac{\hbar^2}{\alpha^2} \langle v \rangle \langle v \rangle_x - \frac{\hbar}{\alpha} D_{qq} \left(\frac{n_x}{n} \langle v \rangle \right)_x + D_{qq}^2 \frac{n_x}{n} \left(\frac{n_x}{n} \right)_x,$$

we are led to

$$\langle v\rangle_t = -\frac{\hbar}{\alpha}\langle v\rangle\langle v\rangle_x - \frac{\alpha}{m\hbar}(V+mU^c)_x - 2\lambda\langle v\rangle$$

$$-\frac{\alpha}{m\hbar}\left[2D_{pq}\ln(n)+\left(1-\frac{4m^2}{\hbar^2}D_{qq}^2\right)Q\right]_x$$

$$+D_{qq}\left(2\frac{n_x}{n}\langle v\rangle_x+\langle v\rangle_{xx}\right)$$

$$-\frac{\alpha}{\hbar}D_{qq}^2\left[\frac{n_x}{n}\left(\frac{n_x}{n}\right)_x-\left(\frac{n_{xx}}{n}\right)_x\right]. \tag{32}$$

Here,

$$Q=-\frac{\hbar^2}{2m}\left[\frac{(\sqrt{n})_{xx}}{\sqrt{n}}\right]=-\frac{\hbar^2}{8m}\left[2\frac{n_{xx}}{n}-\frac{n_x^2}{n^2}\right] \tag{33}$$

is the (enthalpy related) nonlocal quantum potential of Bohm [cf. Eq. (15)] which represents current arising as a result of density gradient effects. Also,

$$U^c(x,t)=\int_0^n \frac{(P_v^c)_{n'}}{n'}dn' \tag{34}$$

is an additional effective potential stemming from the statistical mixture of quantum states. Thus, to the lowest order the quantum open system behaves as an ideal gradient gas (see Ref. [29] for details) subject to frictional and dissipative interactions.

It is well known that quantum potentials play a crucial role in the hydrodynamic description of quantum theory of motion (see, for instance, Refs. [18,31]). In particular, the Bohm potential (33) is basically a field through which the electrons interact with themselves, so that Q_x can be interpreted as a quantum diffusion term yielding a theory which contains both quantum-mechanical confinement effects and tunneling. Bohm's potential has been used, for example, to study wave packet tunneling through barriers. Then, in the picture of the balance of mass and balance of momentum equations (22) and (32), the overall motion of the system under study can be thought of as the motion of a quantum fluid having density $n(x,t)$ and velocity $v(x,t)$, under the influence of the potential $V(x,t)$ augmented by the quantum Bohm potential $Q(x,t)$, the effective potential $U^c(x,t)$ and some viscosity and dissipation forces. The effects due to the appearance of these nonstandard D_{pq} and D_{qq} terms yield significant deviations from what is expected in the purely classical case, as we shall see later on.

C. Derivation of an effective time-dependent Ginzburg-Landau-type equation

This section is devoted to the search for a Schrödinger representation of the quantum hydrodynamic system derived in Sec. II B. This is done by defining the following effective wave function (in modulus-argument representation)

$$\Psi(x,t)=\sqrt{n(x,t)}e^{(i/\alpha)S(x,t)} \tag{35}$$

and considering the velocity field of the fluid flow generated by the wave function (35) to be given by the standard relation

$$\langle v\rangle=\frac{1}{m}S_x. \tag{36}$$

The hydrodynamic to quantum approach stemming from Eqs. (35) and (36) is based on the well-known Madelung transformations [32]. Indeed, the connection between quantum mechanics and classical hydrodynamics was already observed in 1927 by O. Madelung, in the context of the semi-classical approach to nonlinear Schrödinger equations (see Ref. [19] for a recent review). Since then, Madelung-type transformations have been succeedingly used in various scientific fields, e.g., galaxy clustering studies (see Sec. 4 of Ref. [33]) or multistream plasma dynamics [20]. Choosing nonvanishing $n(x,t)$ and nonsingular $S(x,t)$ makes the Madelung transformations meaningful and prevents the associated hydrodynamic equations [cf. Eqs. (38) and (39) below] to become singular. Besides, in Ref. [34] T. Wallstrom observed that the single-valuedness quantization rule

$$\Gamma=\oint_\gamma v(x,t)d\sigma(x)=2k\pi\frac{\hbar}{m} \tag{37}$$

is needed in order to establish the formal equivalence between Madelung and Schrödinger equations, where $k\in\mathbb{Z}$ and γ is any closed loop (in our case, the circulation Γ is required to be an integer multiple of $(2\pi\alpha)/m$). This situation is pointed out to arise only in two or more space dimensions, when removal of the nodal set $\{\Psi=0\}$ of vortex configurations does disconnect the topology. The quantization condition (37) might also be necessary for Madelung-Schrödinger equivalence to hold in 1D whether the topology of the corresponding space is nontrivial.

In the sequel we shall assume for simplicity adequate boundary conditions on the density, the phase, and the total current in order to avoid boundary contributions at infinity along the derivation. In terms of the amplitude and the phase of Ψ, the quantum fluid system consisting of Eqs. (22) and (32) reads

$$n_t+\frac{\hbar}{m\alpha}(nS_x)_x-D_{qq}n_{xx}=0, \tag{38}$$

$$S_t+\frac{\hbar}{2m\alpha}S_x^2=-\frac{\alpha}{\hbar}(V+mU^c)-2\lambda S$$

$$-\frac{\alpha}{\hbar}\left[2D_{pq}\ln(n)+\left(1-\frac{4m^2}{\hbar^2}D_{qq}^2\right)Q\right]$$

$$+D_{qq}\left(2\int_{-\infty}^x \frac{n_x}{n}S_{xx}dx+S_{xx}\right)$$

$$+\frac{\alpha m}{2\hbar}D_{qq}^2\frac{1}{n}\left(2n_{xx}-\frac{n_x^2}{n}\right), \tag{39}$$

which is a dispersive regularization of the quantum Euler system including quantum friction and dissipation corrections. Combining now Eqs. (38) and (39), using the identities (35) and (25) and taking into account the following expressions for the derivatives of the phase

$$S_z = \frac{i\alpha}{2} \frac{1}{n} (n_z - 2\bar{\Psi}\Psi_z),$$ (40)

$$S_{xx} = \frac{i\alpha}{2} \left[\left(\frac{n_x}{n} \right)_x - 2\bar{\Psi} \left(\frac{\Psi_x}{n} \right)_x - 2\frac{|\Psi_x|^2}{n} \right],$$ (41)

where z can be replaced by x or t indistinctively, one gets after simple but lengthy algebra the following exact, nonlinear time-dependent Ginzburg-Landau-type equation

$$i\hbar\Psi_t = \left(-\frac{\hbar^2}{2m} + i\hbar D_{qq} \right)\Psi_{xx} - i\hbar D_{qq}\frac{n_x}{n}\Psi_x$$

$$+ \left(V + mU^c + 2D_{pq}\ln(n) + \eta\int_{-\infty}^x \frac{J}{n}dx \right)\Psi$$

$$- 2mD_{qq}\left[\int_{-\infty}^x \frac{n_x}{n}\left(\frac{J}{n} \right)_x dx \right]\Psi$$

$$+ \frac{i\hbar}{2}D_{qq}\frac{1}{n}\left(2|\Psi_x|^2 + \frac{n_x^2}{n} \right)\Psi.$$ (42)

This equation extends that earliest derived in Ref. [35] by Doebner and Goldin from the simplest form of continuity equation of Fokker-Planck type and looks into the nature of open quantum systems and quantum fluid transitions. In our context (that is, when the diffusion constant is assumed to be D_{qq}), the Doebner-Goldin nonlinear modification to the Schrödinger equation is given by

$$i\hbar\Psi_t = \left(-\frac{\hbar^2}{2m} + i\hbar D_{qq} \right)\Psi_{xx} + \left(V + i\hbar D_{qq}\frac{|\Psi_x|^2}{n} \right)\Psi,$$

which constitutes a reduced model for irreversible and dissipative quantum systems. The presence of a logarithmic nonlinearity in Eq. (42) is justified because we assumed a constant temperature [18]. Otherwise, the Wigner-Fokker-Planck system would not be quantum-mechanically correct (see Ref. [1]) in contrast to the classical situation. Furthermore, $\ln(n)$ can be seen as an approximation of V up to $O(\hbar^2)$ terms when V is assumed to be the Hartree electrostatic potential solving the Poisson equation $V_{xx} = n$ [17]. Some meaningful physical interpretations have been given to the appearance of the potential $\ln(n)$ in the Schrödinger equation. Indeed, it can be understood as the effect of statistical uncertainty or as the potential energy associated with the information encoded in the matter distribution described by the probability density $n(x,t)$ (see [36] and references therein for a wide discussion).

In deriving Eq. (42) we have used the relations (4) and (26), as well as the following identities involving the current density

$$\frac{J}{n} = \frac{\hbar}{m\alpha}S_x,$$

$$\Psi\Psi_x = \frac{n_x}{2} + \frac{im}{\hbar}J.$$ (43)

Equation (42) does incorporate quantum corrections of order $\hbar^2$ and $\hbar^3$ to the nonlinear, frictional (Kostin-Caldeira-Leggett-)Schrödinger equation

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + \left[V + mU^c + \eta\int_{-\infty}^x \frac{J}{n}dx \right]\Psi.$$ (44)

Indeed, Eq. (42) can be rewritten as

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + \left[V + mU^c + \eta\int_{-\infty}^x \frac{J}{n}dx + \Lambda_{\hbar^2} \right]\Psi + \Lambda_{\hbar^3},$$

where the $O(\hbar^2)$ term [cf. Eq. (3)] is given by

$$\Lambda_{\hbar^2} = 2D_{pq}\ln(n)$$

and where [cf. Eq. (25)]

$$\Lambda_{\hbar^3} = -2mD_{qq}\left[\int_{-\infty}^x \frac{n_x}{n}\left(\frac{J}{n} \right)_x dx \right]\Psi$$

$$+ i\hbar D_{qq}\left[\Psi_{xx} - \frac{n_x}{n}\Psi_x + \frac{1}{2n}\left(2|\Psi_x|^2 + \frac{n_x^2}{n} \right)\Psi \right]$$

enters into the equation as a complex potential of order $\hbar^3$. This is in good agreement with the physical interpretation of complex potentials, as they have been used in the literature to simulate dissipative processes and decoherence effects in the transition regions of small quantum devices.

Reciprocally, the quantum hydrodynamic model (38) and (39) is recovered via the Madelung transformation (35) by simply taking real and imaginary parts in both sides of Eq. (42). We note that Eq. (42) is easily checked to be still charge preserving, that is

$$\frac{d}{dt}\left(\int_{\mathbb{R}} |\Psi(x,t)|^2 dx \right) = 0,$$

by just multiplying the equation times $\bar{\Psi}$, integrating against x and taking imaginary parts. Also, following the standard averaging techniques we obtain the following energy functional

$$E[\Psi](t) = \frac{\hbar^2}{2m}\int_{\mathbb{R}} |\Psi_x|^2 dx + \int_{\mathbb{R}} n(V + mU^c)dx$$

$$+ \eta\int_{\mathbb{R}} n\left(\int_{-\infty}^x \frac{J}{n}dx \right)dx + 2D_{pq}\int_{\mathbb{R}} n\ln(n)dx$$

$$- mD_{qq}\int_{\mathbb{R}} n\left[\left(\frac{J}{n} \right)_x + 2\int_{-\infty}^x \frac{n_x}{n}\left(\frac{J}{n} \right)_x dx \right]dx$$ (45)

associated with the solutions of Eq. (42), which of course is not a preserved quantity for the Fermi system (see Refs. [1,2] for details at the Wigner level).

It is significant that the D_{pp} term in Eq. (1) does not contribute to the final form of Eq. (42). This is due to the fact that the moment system is truncated at the level of the current equation, while the D_{pp} contribution is only "visible" at the next level, i.e., that of the kinetic energy equation [cf. Eqs. (10)–(11)]. Indeed, if λ, D_{pq}, and D_{qq} are set to zero (this is the case analyzed in Ref. [1]), Eq. (42) becomes the usual Schrödinger equation with an additional (mixed-state) effective potential $U^c(x,t)$ (as in Ref. [21]). Also, the Caldeira-Leggett model ($D_{pq} = D_{qq} = 0$, see Ref. [12]) reduces to Eq. (44) in this approximation.

Last but not least, Eq. (42) can be rewritten in the traditional form of the Schrödinger equation as

$$i\hbar \Psi_t = -\frac{\hbar^2}{2m} \Psi_{xx} + \frac{i\hbar}{2} D_{qq}\left(\frac{n_{xx}}{n}\right)\Psi$$
$$+ \left(V + mU^c + 2D_{pq}\ln(n) + \eta \int_{-\infty}^{x} \frac{J}{n} dx \right)\Psi$$
$$- mD_{qq}\left[2\int_{-\infty}^{x} \frac{n_x}{n}\left(\frac{J}{n}\right) dx + \left(\frac{J}{n}\right)_x \right]\Psi \quad (46)$$

by observing that

$$\Psi_{xx} - \frac{n_x}{n}\Psi_x + \frac{1}{2n}\left(2|\Psi_x|^2 + \frac{n_x^2}{n} \right)\Psi$$
$$= \left[\frac{n_{xx}}{2n} + \frac{im}{\hbar}\frac{1}{n}\left(J_x - \frac{n_x}{n}J \right) \right]\Psi,$$

which is a simple consequence of the identity (43). Besides the logarithmic nonlinearity, under the "classical" Schrödinger formulation (46), it is more clearly identifiable than in Eq. (42) the presence of a friction term due to the velocity density J/n and a diffusion term due to the diffusive velocity density $D_{qq}(n_x/n)$.

In the especial case of a zero-temperature 1D electron gas, the classical pressure is known to be given by (see, e.g., Ref. [21])

$$P^c(n) = \frac{mv_F^2}{3n_0^2}n^3,$$

where $v_F = (\pi\hbar/2m)n_0$ is the Fermi velocity and n_0 stands for the equilibrium reference density. Accordingly, the effective potential U^c [cf. Eq. (34)] becomes

$$U^c(x,t) = \int_0^n \frac{(P^c)_{n'}}{n'} dn' = \frac{\pi^2\hbar^2}{8m}|\Psi|^4.$$

The resulting configuration is similar (see Ref. [37]) to that standing for a 1D mean-field Boson gas in the dilute limit and strong coupling condition. Another interesting example arises in the 3D case, where the multidimensional analog of Eq. (42) now incorporates an additional effective potential

$U^c(x,t)$ which is proportional to $|\Psi|^{4/3}$ (see the Appendix). This potential comes out from an X^α approximation of the Hartree-Fock exchange-correlation potential (Thomas-Fermi approach) when the charge density variation is not too rapid, and is commonly used in practical numerical calculations.

III. THE STOCHASTIC APPROACH

This section is devoted to the derivation of a Ginzburg-Landau-type model from the Wigner-Fokker-Planck equation, still accounting for quantum friction and dissipation effects. The resulting Ginzburg-Landau type equation [cf. Eq. (58) below] incorporates a logarithmic nonlinearity to a particular nonlinear Schrödinger equation of the Doebner-Goldin class [35]. The way to proceed now lies on a different perspective. Actually, we shall use stochastic techniques based on consideration of forward and backward velocities associated with quantum Brownian motion and time reversal invariance. Although seemingly simpler than Eq. (46), the Eq. (58) retains terms up to order $O(\hbar^4)$.

We now assume that the Wigner-Fokker-Planck equation (1) and (2) [or more precisely the moment system consisting of Eqs. (16) and (18)–(20)] is associated with a stochastic process. Indeed, assuming Brownian motion of the electron ensemble (for instance, produced by the interaction of the electrons with a ionic lattice), the electrons follow the paths of the stochastic Langevin equation (Newton's law)

$$mx'' - V_x = \mathcal{F}(t). \quad (47)$$

Here, the effect of the coupling between the system and the thermal bath is modeled by the stochastic force

$$\mathcal{F}(t) = \Gamma(t) - 2\lambda x'(t),$$

which consists of a mean (friction) force proportional to the velocity plus a fluctuating term $\Gamma(t)$, $\Gamma(t)$ being a white noise random force with independent, identically Gaussian distributed processes of (constant) variance $2\lambda m^2\theta$, with $\theta = (k_B T)/m$. Applying now Ito's equation for the probability density associated with the solutions of Eq. (47) yields the well-known Vlasov-Fokker-Planck equation

$$f_t + \xi f_x - \frac{1}{m}V_x f_\xi = 2\lambda(\xi f + \theta f_\xi)_\xi \quad (48)$$

(see, for example, the introduction of Ref. [38], where this derivation was sketched for a quantum Fokker-Planck relaxation model). The quantum-mechanical analog of Eq. (48) is our Wigner-Fokker-Planck equation (1) and (2) (see Ref. [1] for details). Actually, in the classical limit $\hbar \to 0$ we have $D_{qq} \to 0, D_{pq} \to 0$ and formally

$$L_{QFP}W \to \frac{D_{pp}}{m^2}W_{\xi\xi} + 2\lambda(\xi W)_\xi,$$

$$\Theta[V]W \to -\frac{1}{m}V_x W_\xi,$$

such that the classical Vlasov-Fokker-Planck equation (48) is recovered.

In this picture, Eq. (16) admits a classical interpretation in terms of Nelsonian stochastic mechanics [39] (see also Refs. [40–42] and references therein). This theory, initiated in 1952 by Fénies [43], is intended to give a description of quantum mechanics in terms of classical probability densities for particles undergoing Brownian motion with diffusive interactions. In this framework, E. Nelson showed in Ref. [39] that the evolution of a particle subject to nondissipative Brownian motion is equivalent (in the sense of its probability and current density) to that described by the Schrödinger equation (see also Refs. [42,44]). Fruitful applications of this stochastic picture are in order nowadays, see, for instance, Refs. [22,30] where the nonlinear dynamics of particles in accelerators has been studied in the framework of a stochastic-hydrodynamic model for the collective motion of a particle beam. We shall make here some extensions aiming at the frictional, dissipative case.

Under the Brownian motion assumption, the particles are subject to the action of forward and backward velocity fields u_+ and $u_- = u_+ - 2u_o$, respectively, which enter into the density equation (16) as

$$n_t + (nu_\pm)_x = \pm D_{qq} n_{xx}. \tag{49}$$

Here, u_o denotes the so-called osmotic velocity defined by

$$u_o := D_{qq} \frac{n_x}{n} \tag{50}$$

according to Fick's law, which sets the exact balance between the osmotic current nu_o and the diffusion current $D_{qq}n_x$ and somehow controls the degree of stochasticity of the process by informing us about how much nondifferentiable the random trajectories of the particles are. Now, summing up both forward and backward Fokker-Planck equations in Eq. (49) and introducing the current velocity

$$v := \frac{u_+ + u_-}{2} = u_+ - u_o = u_+ - D_{qq} \frac{n_x}{n},$$

we recover the continuity equation $n_t + (nv)_x = 0$. On the other hand, by defining the mean backward derivative of the forward velocity as

$$(\mathcal{D}_- u_+)(x,t) := (u_+)_t + u_-(u_+)_x - D_{qq}(u_+)_{xx},$$

Eq. (18) can be rewritten (for u_+) as

$$\mathcal{D}_- u_+ = -\frac{1}{m}V_x - \frac{1}{n}(P_{u_+})_x - 2\lambda u_+ - \frac{2}{m}D_{pq}\frac{n_x}{n}. \tag{51}$$

We now perform time inversion in Eq. (51) according to the following transformations for the time variable, the backward and forward velocities and the mean backward and forward derivatives [40]:

$$t \mapsto -t, \quad z_t \mapsto -z_t, \quad u_\pm \mapsto -u_\mp, \quad \mathcal{D}_\pm \mapsto -\mathcal{D}_\mp.$$

Since the internal stress tensor P_{u_+} is a dynamic characteristic of motion, its divergence changes sign under time inversion. Accordingly, after time inversion Eq. (51) becomes

$$\mathcal{D}_+ u_- = -\frac{1}{m}V_x + \frac{1}{n}(P_{u_+})_x + 2\lambda u_- - \frac{2}{m}D_{pq}\frac{n_x}{n}, \tag{52}$$

with

$$(\mathcal{D}_+ u_-)(x,t) := (u_-)_t + u_+(u_-)_x + D_{qq}(u_-)_{xx}.$$

A similar procedure can be followed in Ref. [22]. Subtracting Eq. (52) from Eq. (51) yields

$$(u_o)_t + v(u_o)_x = v_x u_o + D_{qq}v_{xx} - 2\lambda v - \frac{1}{n}(P_{u_+})_x, \tag{53}$$

or equivalently the following expression for the pressure field

$$P_{u_+} = 2D_{qq}nv_x - 2\lambda \int_{-\infty}^{x} j\,dx, \tag{54}$$

where $j = nv$ and we used Eq. (50).

We then sum up Eqs. (51) and (52). We obtain the following frictional version of Nelson's stochastic generalization of Newton's law

$$v_t + vv_x = -\frac{1}{m}V_x - 2\left(\lambda D_{qq} + \frac{D_{pq}}{m}\right)\frac{n_x}{n}$$
$$- D_{qq}^2\left[\frac{n_x}{n}\left(\frac{n_x}{n}\right)_x - \left(\frac{n_{xx}}{n}\right)_x\right]. \tag{55}$$

Invoking the statistical velocity averages introduced in Section II B, we get from Eq. (55) the following transport equation for the momentum density

$$\langle v \rangle_t + \frac{\hbar}{\alpha}\langle v \rangle \langle v \rangle_x = -\frac{\alpha}{m\hbar}V_x - \frac{2\alpha}{\hbar}\left(\lambda D_{qq} + \frac{D_{pq}}{m}\right)\frac{n_x}{n}$$
$$- \frac{\alpha}{\hbar}D_{qq}^2\left[\frac{n_x}{n}\left(\frac{n_x}{n}\right)_x - \left(\frac{n_{xx}}{n}\right)_x\right] - D_{qq}\langle v \rangle_{xx}$$
$$- \frac{4m\alpha}{\hbar^3}D_{qq}^2 Q_x, \tag{56}$$

α still standing for Eq. (26). The Schrödinger-like equation associated with Eq. (56) comes out by using the Madelung wave function (35). Indeed, identifying the average momentum $\langle v \rangle$ with a multiple of the phase gradient [cf. Eq. (36)] and taking into account the relations (40) and (41), we find the following Ginzburg-Landau-type equation

$$i\hbar\Psi_t=\left(-\frac{\hbar^2}{2m}-i\hbar D_{qq}\right)\Psi_{xx}+i\hbar D_{qq}\frac{n_x}{n}\Psi_x$$

$$+\left[V-\left(1-\frac{8m^2}{\hbar^2}D_{qq}^2\right)Q\right]\Psi+(2D_{pq}+\eta D_{qq})$$

$$\times\ln(n)\Psi-i\hbar D_{qq}\left(\frac{4m}{\hbar^2}Q+\frac{|\Psi_x|^2}{n}\right)\Psi,\qquad(57)$$

by performing analogous calculations as those for the hydrodynamic approach. Here, $Q(x,t)$ denotes again the quantum potential of Bohm introduced in Eq. (33). As done in Sec. II C, Eq. (57) may be now recast in the traditional form of Schrödinger's equation augmented by a complex potential. We find

$$i\hbar\Psi_t=-\frac{\hbar^2}{2m}\Psi_{xx}+i\frac{\hbar}{2}D_{qq}\frac{n_{xx}}{n}\Psi$$

$$+\left[V-\left(1-\frac{8m^2}{\hbar^2}D_{qq}^2\right)Q\right]\Psi$$

$$+(2D_{pq}+\eta D_{qq})\ln(n)\Psi+mD_{qq}\left(\frac{J}{n}\right)_x\Psi.\quad(58)$$

Note that an additional $O(\hbar^4)$ quantum correction appears involving the Bohm potential.

It is clear again that Eq. (58) preserves mass. Also, the energy functional associated with the wave function solutions of Eq. (58) is now given by

$$E[\Psi]=\frac{\hbar^2}{2m}\int_{\mathbb{R}}|\Psi_x|^2dx+\int_{\mathbb{R}}Vndx+(2D_{pq}+\eta D_{qq})$$

$$\times\int_{\mathbb{R}}\ln(n)ndx-mD_{qq}\int_{\mathbb{R}}\left(\frac{n_x}{n}\right)Jdx$$

$$-\left(\frac{\hbar^2}{8m}-mD_{qq}^2\right)\int_{\mathbb{R}}\frac{n_x^2}{n}dx.$$

IV. ABOUT STATIONARY SOLUTIONS

In this section we deal with the problem of existence of stationary solutions to the nonlinear Schrödinger equations (46) and (58) derived above. In this direction, we analyze the wave function profiles leading to stationarity and make some remarks concerning the dynamics of some physical observables in the force-free case.

A. Stationary solutions to Eq. (46)

We shall first find solutions satisfying the stationarity condition $n_t=0$, which straightforwardly leads to

$$J=D_{qq}n_x\qquad(59)$$

via Eq. (16). Hence, Eq. (46) becomes

$$i\hbar\Psi_t=-\frac{\hbar^2}{2m}\Psi_{xx}+(V+mU^c)\Psi+(2D_{pq}+\eta D_{qq})\ln(n)\Psi$$

$$+\left(\frac{i\hbar}{2}D_{qq}-mD_{qq}^2\right)\frac{n_{xx}}{n}\Psi.\qquad(60)$$

Consider the following ansatz

$$\Psi(x,t)=u(x)e^{iS(x,t)},\qquad(61)$$

where $u(x)=\sqrt{n(x)}$ and $S(x,t)$ are real functions. Then, from Eq. (59) and the definition of the electric current $J=(\hbar/m)\mathrm{Im}(\overline{\Psi}\Psi_x)$ one gets $(\hbar/m)nS_x=D_{qq}n_x$, thus

$$S(x,t)=\frac{m}{\hbar}D_{qq}\ln(n(x))-\Omega(t),$$

$\Omega(t)$ being (in principle) an arbitrary function of time. Consequently, the stationary profile (61) now reads

$$\Psi(x,t)=u(x)\exp\left\{2i\frac{m}{\hbar}D_{qq}\ln(u(x))-i\Omega(t)\right\}.\quad(62)$$

Inserting Eq. (62) into Eq. (60) and choosing $\Omega(t)=\omega t$ for consistency we find

$$\hbar\omega u=\left(-\frac{\hbar^2}{2m}-2mD_{qq}^2\right)u''$$

$$+[V+mU^c+2(2D_{pq}+\eta D_{qq})\ln(u)]u.\quad(63)$$

Here, primes denote x derivatives. We then claim that the ansatz profile (62) solves Eq. (46) iff $u(x)$ obeys the nonlinear stationary Schrödinger equation (63).

We next discuss the simplest case $V=U^c=0$. First, we point out that the constant function

$$u\equiv\exp\left\{\frac{\hbar\omega}{2(2D_{pq}+\eta D_{qq})}\right\}\qquad(64)$$

is a particular solution of Eq. (63), hence

$$\Psi=\exp\left\{\frac{\hbar\omega}{2(2D_{pq}+\eta D_{qq})}+i\omega\left(\frac{mD_{qq}}{2D_{pq}+\eta D_{qq}}-t\right)\right\}$$

is a stationary solution of the force-free nonlinear Schrödinger equation

$$i\hbar\Psi_t=-\frac{\hbar^2}{2m}\Psi_{xx}+\frac{i\hbar}{2}D_{qq}\left(\frac{n_{xx}}{n}\right)\Psi$$

$$+\left(2D_{pq}\ln(n)+\eta\int_{-\infty}^x\frac{J}{n}dx\right)\Psi$$

$$-mD_{qq}\left[2\int_{-\infty}^x\frac{n_x}{n}\left(\frac{J}{n}\right)dx+\left(\frac{J}{n}\right)_x\right]\Psi.\quad(65)$$

In order to find other nontrivial solutions we just multiply Eq. (63) by u' and integrate against x, then we obtain

$$F(u,u')=k\in\mathbb{R},\qquad(66)$$

with

$$F(u,u')=(u')^2-\frac{4m(2D_{pq}+\eta D_{qq})}{\hbar^2+4m^2D_{qq}^2}\ln(u)u^2$$

$$+\frac{2m(\hbar\omega+2D_{pq}+\eta D_{qq})}{\hbar^2+4m^2D_{qq}^2}u^2. \tag{67}$$

Equations (66) and (67) are easily shown to have a saddle point at $(u_0,0)$, where u_0 is given by formula (64). Indeed, its phase portrait (for typical values of the diffusion constants) is as shown in Fig. 1(a) and 1(b). A detailed scrutiny of the phase portrait shows that the only "mathematically meaningful" solutions to Eq. (66) and (67) [thus to Eq. (63)] are those at the left of the saddle point [depicted in Fig. 2(d)] when extension by zero of $u\ln(u)$ is assumed by continuity, in the sense that they do belong to all $L^p(\mathbb{R})$ with $1\leqslant p\leqslant\infty$. This opens the way to the analysis of asymptotic behavior and orbital stability in future work. On the other hand, solutions corresponding to the branches above and below the saddle point are all unbounded. Also, the only solutions originally defined on the whole line (before extension by zero) are those corresponding to the branches allocated at the right of the saddle point, but all of them are not bounded neither decaying. A typical example of the last class of solutions is easily computed by assuming $k=0$ in Eq. (66). In doing so we get

$$u=\exp\left\{\frac{1}{2}\left(1+\frac{\hbar\omega}{2D_{pq}+\eta D_{qq}}\right)+\frac{m(2D_{pq}+\eta D_{qq})}{\hbar^2+4m^2D_{qq}^2}x^2\right\}$$

and the associated stationary state

$$\Psi=\exp\left\{\frac{1}{2}\left(1+\frac{\hbar\omega}{2D_{pq}+\eta D_{qq}}\right)+\frac{m(2D_{pq}+\eta D_{qq})}{\hbar^2+4m^2D_{qq}^2}x^2\right.$$

$$+i\frac{m}{\hbar}D_{qq}\left(1+\frac{\hbar\omega}{2D_{pq}+\eta D_{qq}}+\frac{2m(2D_{pq}+\eta D_{qq})}{\hbar^2+4m^2D_{qq}^2}x^2\right)$$

$$\left.-i\omega t\right\}$$

which solves Eq. (65).

Another way of finding stationary solutions consists of assuming $J\equiv0$. Under this assumption, Eq. (46) becomes

$$i\hbar\Psi_t=-\frac{\hbar^2}{2m}\Psi_{xx}+\frac{i\hbar}{2}D_{qq}\left(\frac{n_{xx}}{n}\right)\Psi$$

$$+[V+mU^c+2D_{pq}\ln(n)]\Psi. \tag{68}$$

As in the previous case, taking $u(x,t)=\sqrt{n(x,t)}$ yields

$$u_t=D_{qq}\left(\frac{u_x^2}{u}+u_{xx}\right). \tag{69}$$

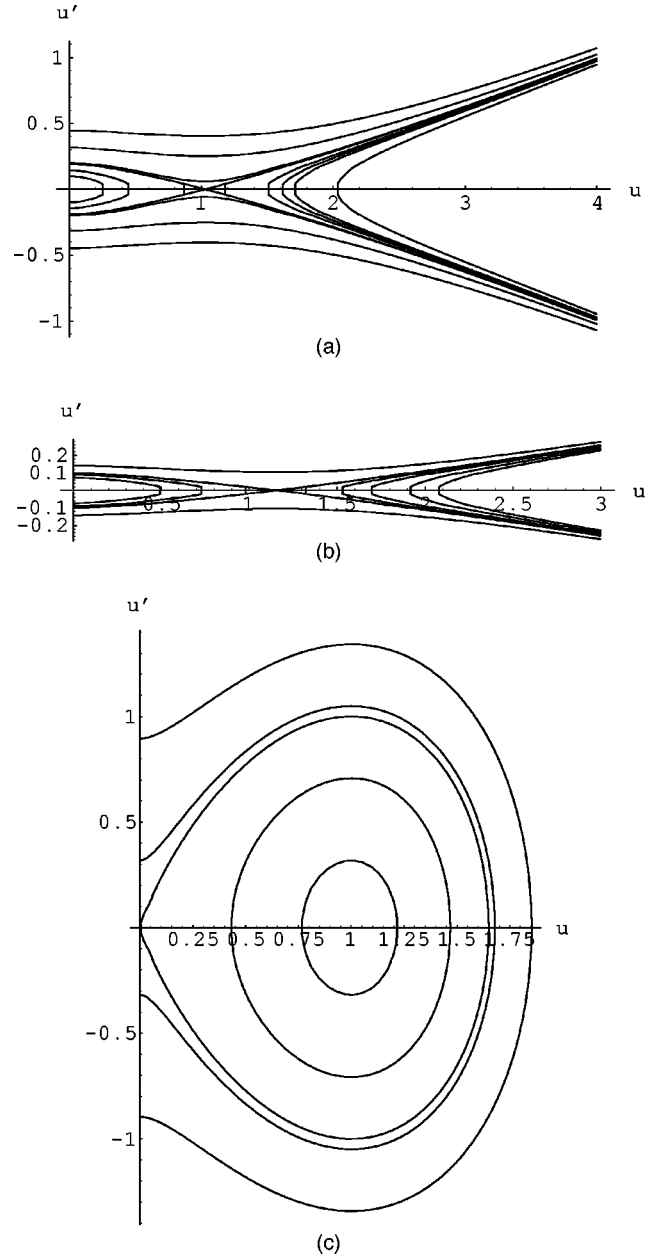Now, considering the wave function profile

(a)

(b)

(c)

FIG. 1. Top to bottom: (a) Phase portrait of Eq. (63) for typical values of the diffusion coefficients of Dekker's phenomenology in the high-temperature regime $T=2$ and $\Omega=1$, in units such that the Boltzmann constant is unity. The damping constant $\lambda=0.2$. (b) Same as (a), except $T=1$, $\Omega=0.5$. The damping constant $\lambda=0.05$ and $\omega=0.001$. (c) The counterpart of Eq. (63) [with the physical constants set in (a)] with logarithmic nonlinearity of Byalinicki-Virula-Mycielski type, i.e., with a minus sign in front of the logarithmic term.

$$\Psi(x,t)=u(x,t)e^{iS(t)} \tag{70}$$

and inserting it into Eq. (68) one easily gets

$$Eu=-\frac{\hbar^2}{2m}u_{xx}+[V+mU^c+4D_{pq}\ln(u)]u, \tag{71}$$
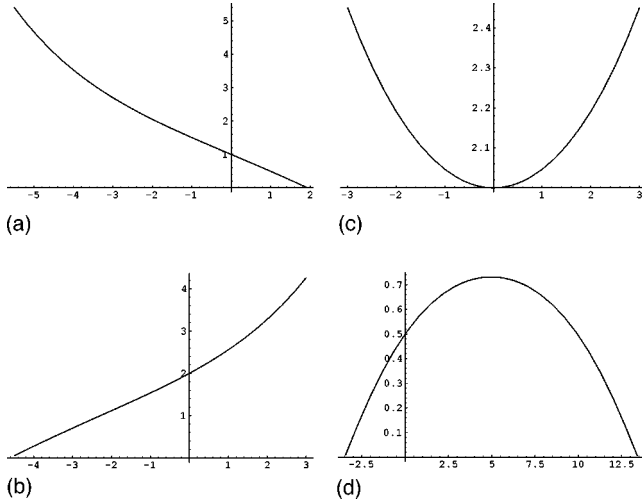
where we have used Eq. (69) and denoted

FIG. 2. The four types of solutions $u(x)$ to Eq. (63) [with the physical constants set in Fig. 1(a)]. Top to bottom: (a) Initial data $(u(0),u'(0))=(1,-0.5)$, (b) Initial data (2,0.5), (c) Initial data (2,0), and (d) Initial data (0.5,0.1).

$$E(t)=-\hbar\frac{dS}{dt}. \tag{72}$$

As consequence, the ansatz (70) is a solution of Eq. (46) iff $u(x,t)$ solves Eqs. (69) and (71) with $E(t)$ given by Eq. (72). Finally, from Eq. (71) it is a simple matter to show that the stationary states given by Eq. (70) satisfy the following non-linear stationary Schrödinger equation

$$E\Psi=-\frac{\hbar^2}{2m}\Psi_{xx}+[V+mU^c+2D_{pq}\ln(n)]\Psi.$$

By making the particular choice

$$V=U^c=0, \quad D_{qq}=0, \tag{73}$$

one can easily recognize from Eq. (68) [or Eq. (60)] the well-known logarithmic Schrödinger equation. This equation has been widely analyzed by several authors, e.g., Cazenave [45,46] (in the case that the logarithmic term is preceded by a minus sign) and Cid and Dolbeault [47]. In particular, some dispersion estimates and stability properties have been investigated. Indeed, if Eq. (73) is assumed and in addition the following regularities

$$\Psi_0\in H^1(\mathbb{R}), \quad |x|\Psi_0\in L^2(\mathbb{R}),$$

$$|x|^2n,n\ln(n)\in C^0[\mathbb{R}^+;L^1(\mathbb{R})]$$

are satisfied, then the dispersion property

$$\int_I n(x,t)dx\leqslant O\left(\frac{1}{\ln(t)}\right)$$

was shown in Ref. [47] to hold for evolutionary solutions as $t\rightarrow+\infty$, for all I bounded open set in $\mathbb{R}$. Here, $\Psi_0=\Psi(x,t=0)$ denotes the corresponding initial data. However, if we assume Eq. (73) in our current setting (vanishing current) we

are actually led to stationary solutions of the type discussed above ($n_t=0$, see Fig. 2) as sheds from Eq. (69). This means that the Dolbeault-Cid dispersion result stated before does not apply in our situation, when Eq. (68) is reduced to the logarithmic Schrödinger equation.

On the other hand, well-posedness and stability results were also shown in Refs. [45,46] for the logarithmic Schrödinger equation, where the logarithmic nonlinearity was considered with opposite sign to that explored in this paper. Nevertheless, this fact drastically changes the dynamics of the system [see Fig. 1(c)]. For instance, in this situation Gaussons are admitted as amplitude profiles for stationary solutions [26]. In fact, the radial Gausson was shown in [45] to be orbitally stable under radial perturbations. The single sign choice for the logarithmic term first made in [26] and later continued in [45,46] was owing to the fact that the other sign led to an energy functional which was not bounded from below. However, the positive sign for the logarithmic nonlinearity was physically justified in Ref. [48] as representing a diffusion force within the context of stochastic quantum mechanics. Indeed, by considering slowly varying profiles in the absence of external forces [as those shown in Fig. 2(d)] we guess that the kinetic contribution $\int_{\mathbb{R}}(u_x)^2dx$ is negligible, so that the effective energy operator (45) may be now rewritten as

$$E[u]=2D_{pq}\int_{\mathbb{R}}u^2\ln(u^2)dx. \tag{74}$$

This expression is not bounded from below. However, whether the density profile $u(x)$ has support over a domain I with finite measure in configuration space, then it is a simple matter to check that the energy functional (74) satisfies

$$E[u]\geqslant-\frac{2D_{pq}}{e}|I|,$$

$|I|$ denoting the Lebesgue measure of I. If in addition the normalization constraint $\int_{\mathbb{R}}n(x)dx=1$ is assumed to hold, then minimization of the functional

$$E_\mu[u]=\int_I\left[u^2\ln(u^2)+\mu\left(u^2-\frac{1}{|I|}\right)\right]dx$$

leads to the minimizer $u\equiv|I|^{-1/2}$, where we used the Lagrange multiplier μ. Therefore, the following stronger bound on (74)

$$E[u]\geqslant-2D_{pq}\ln(|I|)$$

is deduced. A similar argument was carried out by Davidson in the latest reference of those cited in Ref. [48] to conclude that the usual assumption about the sign of the logarithmic term made in Ref. [26] is not the only reasonable possibility and that a sensible theory can be developed with the opposite sign as well.

B. Stationary solutions to Eq. (58)

We now study the existence of the following type of stationary profiles

$$\Psi(x,t) = e^{R(x,t) + iS(x,t)} \qquad (75)$$

as wave function solutions to Eq. (58). By inserting the ansatz (75) into Eq. (58) and splitting into real and imaginary parts we are led to the following system of nonlinear coupled equations

$$R_t = -\frac{\hbar^2}{2m} S_{xx} - \frac{\hbar}{m} R_x S_x + D_{qq} R_{xx} + 2 D_{qq} R_x^2, \qquad (76)$$

$$S_t = -\frac{1}{\hbar} V - \frac{2}{\hbar}(2D_{pq} + \eta D_{qq}) R - D_{qq} S_{xx} + \frac{4m}{\hbar} D_{qq}^2 R_{xx}$$

$$+ \frac{4m}{\hbar} D_{qq}^2 R_x^2 - \frac{\hbar}{2m} S_x^2. \qquad (77)$$

Again we are looking for stationary states satisfying $n_t = 0$, or equivalently $R_t = 0$. Under this assumption, Eq. (76) is reduced to

$$\left(D_{qq} R - \frac{\hbar^2}{2m} S \right)_{xx} + 2R_x \left(D_{qq} R - \frac{\hbar}{2m} S \right)_x = 0. \qquad (78)$$

We first assume $R(x) \equiv \beta \in \mathbb{R}$, thus Eq. (78) yields

$$S(x,t) = kx + h(t), \quad k \in \mathbb{R},$$

$h(t)$ being an arbitrary function of time. Denoting $\psi_0 = e^\beta$ and choosing $h(t) = -\omega t$ with $\omega \in \mathbb{R}$, we find the following plane wave solutions

$$\Psi(x,t) = \psi_0 e^{i(kx - \omega t)}, \quad \psi_0 \in \mathbb{R},$$

to the field-free ($V \equiv 0$) Eq. (58), which constitute a particular case of the profiles (75). Finally, the following expression for the k-dependence of ω

$$\omega(k) = \frac{\hbar}{2m} k^2 + \frac{2\beta}{\hbar}(2D_{pq} + \eta D_{qq})$$

is deduced from Eq. (77).

We now investigate the existence of other nontrivial stationary solutions to Eq. (58). For that, we first consider the following ansatz for the phase of the wave function

$$S(x,t) = \frac{2m}{\hbar}(D_{qq} R(x) - \omega t). \qquad (79)$$

It is a simple matter to check that the functions in Eq. (79) solve Eq. (78). Moreover, from Eq. (77) we find the following phase equation

$$\omega = -D_{qq}^2 [R'' + (R')^2] + \frac{1}{2m} V + \frac{1}{m}(2D_{pq} + \eta D_{qq}) R. \qquad (80)$$

Now, if $R(x)$ is a solution of Eq. (80) then the amplitude function

$$u(x) = e^{R(x)}$$

satisfies

$$2m\omega u = -2mD_{qq}^2 u'' + [V + 2(2D_{pq} + \eta D_{qq}) \ln(u)] u. \qquad (81)$$

Hence, from the solutions $u(x)$ of Eq. (81) we get the following stationary solutions of Eq. (58)

$$\Psi(x,t) = u(x) \exp\left\{ i\frac{2m}{\hbar} [D_{qq} \ln(u(x)) - \omega t] \right\}.$$

In the particular case $D_{qq} = 0$ we find that

$$\Psi(x,t) = \psi_0 \exp\left\{ R(x) - i\frac{2m}{\hbar}\omega t \right\}, \quad \psi_0 \in \mathbb{R},$$

solves Eq. (58), where $R(x)$ and ω are connected through

$$\omega = \frac{1}{2m} V + \frac{2}{m} D_{pq} R$$

as sheds from Eq. (80). Besides, if $V \equiv 0$ then the simplest (constant) solution of Eq. (81) is

$$u \equiv \exp\left\{ \frac{m\omega}{2D_{pq} + \eta D_{qq}} \right\},$$

therefore we may claim that

$$\Psi(x,t) = \exp\left\{ \frac{m\omega}{2D_{pq} + \eta D_{qq}} \left[1 + \frac{2i}{\hbar} H(t) \right] \right\}$$

is a particular stationary solution of Eq. (58), with

$$H(t) = mD_{qq} - (2D_{pq} + \eta D_{qq}) t.$$

We also notice that Eq. (81) is structurally equivalent to Eq. (63), hence the behavior of its solutions is just as explained in the preceding section [cf. Figs. 1(a), 1(b), and 2].

We finally investigate the stationary solutions stemming from the vanishing of the current flux ($J \equiv 0$). In this case it is straightforward to verify that the position density satisfies the following heat equation

$$n_t = D_{qq} n_{xx}$$

whose solution is given by

$$n(x,t) = \frac{1}{\sqrt{4\pi t}} \int_{\mathbb{R}} n_0(y) e^{-[(x - \sqrt{D_{qq}} y)^2 / 4D_{qq} t]} dy$$

for given initial data $n_0(x) = n(x, t=0)$. Equivalently, $u = \sqrt{n}$ satisfies

$$u_t = D_{qq} \left(\frac{u_x^2}{u} + u_{xx} \right). \qquad (82)$$

On the other hand, Eq. (58) is now reduced to

$$i\hbar\Psi_t = -\frac{\hbar^2}{2m}\Psi_{xx} + \frac{i\hbar}{2}D_{qq}\left(\frac{n_{xx}}{n}\right)\Psi + [V + (2D_{pq} + \eta D_{qq})$$

$$\times \ln(n)]\Psi + \left(\frac{\hbar^2}{4m} - 2mD_{qq}^2\right)\left(\frac{n_{xx}}{n} - \frac{n_x^2}{2n^2}\right)\Psi. \quad (83)$$

Consider the following ansatz

$$\Psi(x,t) = u(x,t)e^{i\Omega(t)}. \quad (84)$$

Inserting this profile into Eq. (83) and using Eq. (82) yields

$$Eu = -4mD_{qq}^2 u_{xx} + [V + 2(2D_{pq} + \eta D_{qq})\ln(u)]u, \quad (85)$$

where $E(t)$ is given by Eq. (72). Hence, the wave function (84) is a solution of Eq. (58) provided that $E(t)$ and $u(x,t)$ satisfy Eq. (85), which again fits the dynamics of the logarithmic Schrödinger equation discussed in the previous section.

From Eqs. (79) and (80) it is also possible to deduce the existence of solitary waves for the force-free case. Indeed, let us define

$$\tilde{R}(x,t) = R(x - vt, t),$$

$$\tilde{S}(x,t) = S(x - vt, t) + \frac{m}{\hbar}vx - \frac{m}{2\hbar}v^2t.$$

It is a simple matter to check that the pair $(\tilde{R}, \tilde{S})$ solves Eqs. (76) and (77). Hence, the wave function profile (75) is now written as

$$\Psi(x,t) = \psi_0 \exp\left\{R(x - vt) + \frac{im}{\hbar}\left[2D_{qq}R(x - vt) - 2\omega t + vx\right.\right.$$

$$\left.\left. - \frac{v^2}{2}t\right]\right\}, \quad \psi_0 \in \mathbb{R},$$

where R is a solution of Eq. (80). Accordingly, the position density reads

$$n(x,t) = \psi_0^2 e^{2R(x - vt)}$$

yielding solitary waves which propagate with constant velocity v without changing their shape.

V. SUMMARY AND CONCLUSIONS

In this paper we have derived two nonlinear Schrödinger-type models for an electron gas interacting dissipatively with a reservoir, which correspond to a hydrodynamic and a stochastic approach, respectively. We start from the one-dimensional Wigner-Fokker-Planck system, which is an extension of the well-known Caldeira-Leggett master equation including quantum diffusion in the position direction. This equation preserves mass, but neither moment nor energy are conserved along the evolution.

In the hydrodynamic approach we show that, under plausible physical assumptions on the (mixed-state) classical pressure and the amplitude of the waves, which allow for moments closure, the quantum-mechanical nonlinear dynamics of the dissipative process is reduced to a nonlinear complex Schrödinger equation including a friction term of Kostin-type [27], a Drude type correction (represented by a logarithmic nonlinearity with opposite sign to that derived in Ref. [26]) and a $O(\hbar^3)$ complex, diffusive correction which allocates the problem in a generalized Ginzburg-Landau setting. A nonlinear equation of the same type including $O(\hbar^4)$ diffusive corrections is obtained through the Nelsonian stochastic approach, although in this case the pressure tensor is fully identified via some simple algebra involving the forward and backward mean velocities associated with the process. We thus conclude that physical situations described by the quantum Fokker-Planck equation (1) and (2), e.g., coherences and tunneling phenomena, in which statistical fluctuations may occur allowing for dissipative effects, can be modeled by Eqs. (42) or (57) [equivalently, by Eqs. (46) or (58)]. Finally, an exhaustive description of how stationarity occurs for both models has been carried out. The most remarkable point in this context is the existence of solitoniclike density profiles, which also have mathematical interest because of their regularity properties in Sobolev spaces.

The analysis of some distinctive features of these equations deserves of further research, such as the existence of other particular solutions (for instance, the study of Gaussian shape, soliton or self-similar solutions), the dispersive behavior of solutions in the long time as well as the stability properties of the L^p stationary solutions, the analysis of the large friction regime leading to the quantum Smoluchowski equation in the Wigner picture or the rigorous inviscid limit which allows to recover the frictional, nondissipative Schrödinger equation, among others. Also, a comparative qualitative and numerical analysis for both hydrodynamic and stochastic dissipative Schrödinger models is an interesting subject of future work.

APPENDIX: SOME REMARKS ON THE MULTIDIMENSIONAL DERIVATIONS

The 3D version of the stochastic approach carried out in Sec. III is straightforward by following the 1D computations outlined above in the multidimensional case. It is significant the fact that the 3D analog of Eq. (53) implies

$$\sum_{i=1}^{3}\frac{\partial P_{ik}}{\partial x_i} = D_{qq}\sum_{i=1}^{3}\frac{\partial}{\partial x_i}\left[n\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right)\right] - 2\lambda n v_k,$$

hence the 3D version of Eq. (54) is written in terms of the following rank two tensor

$$P_{ik} = D_{qq}n\left(\frac{\partial v_i}{\partial x_k} + \frac{\partial v_k}{\partial x_i}\right) - \frac{2}{3}\lambda \int_{-\infty}^{x_i} j_k dx_i. \qquad (A1)$$

This identity incorporates a friction term to the classical expression of the rate of strain tensor in viscous fluid dynamics.

On the other hand, things are not so simple for the 3D calculations leading to the hydrodynamic Schrödinger approach. Actually, the right-hand side of the (kth component of the) following Hamilton-Jacobi-Madelung-type equation [cf. Eq. (39)] ruling the evolution of $\nabla_x S$

$$\frac{\partial}{\partial t}\left(\frac{\partial S}{\partial x_k}\right) + \frac{\hbar}{m\alpha}\sum_{i=1}^{3}\frac{\partial S}{\partial x_i}\frac{\partial^2 S}{\partial x_i \partial x_k}$$

$$= -\frac{\alpha}{\hbar}\frac{\partial V}{\partial x_k} - \frac{m\alpha}{\hbar}\sum_{i=1}^{3}\frac{\partial U_{ik}^c}{\partial x_i} - \frac{2\alpha}{\hbar}D_{pq}\frac{1}{n}\frac{\partial n}{\partial x_k}$$

$$- 2\lambda\frac{\partial S}{\partial x_k} + \frac{D_{qq}}{n}\sum_{i=1}^{3}\left[2\frac{\partial n}{\partial x_i}\frac{\partial^2 S}{\partial x_i \partial x_k} + n\frac{\partial^3 S}{\partial x_i^2 \partial x_k}\right.$$

$$- \frac{\hbar}{2}\left(\frac{1}{n}\frac{\partial^2 n}{\partial x_i^2}\frac{\partial n}{\partial x_k} + \frac{1}{n}\frac{\partial n}{\partial x_i}\frac{\partial^2 n}{\partial x_i \partial x_k}\right.$$

$$\left.\left. - \frac{1}{n^2}\left(\frac{\partial n}{\partial x_i}\right)^2\frac{\partial n}{\partial x_k} - \frac{\partial^3 n}{\partial x_i^2 \partial x_k}\right)\right] \qquad (A2)$$

is no longer irrotational, hence Eq. (A2) is not consistent from a mathematical point of view. Indeed, a sufficient condition for consistency is that $(\nabla_x u_o \nabla_x \langle v \rangle)$ be a symmetric matrix.

Finally, in the hydrodynamic approach the classical component of the tensor pressure

$$P_{ik} = \int_{\mathbb{R}} \xi_i \xi_k W d\xi - nv_i v_k$$

is given by

$$P_{ik}^c = P_{ik}^c(n) = \frac{\hbar^2}{\alpha^2}n(\langle v_i v_k \rangle - \langle v_i \rangle \langle v_k \rangle).$$

We now define

$$U_{ik}^c(n) := \int_0^n \frac{1}{n'}\frac{\partial P_{ik}^c}{\partial n'}dn'.$$

In the lowest-order isotropic approximation, the local hydrostatic pressure of the electron gas is known to be proportional to $n^{5/3}I$, where I denotes the identity matrix. This expression actually satisfies the condition of adiabatic transformation for an ideal gas. We have

$$P_{ik}^c \approx n^{5/3}\delta_{ik},$$

thus

$$\sum_{i=1}^{3}\frac{\partial U_{ik}^c}{\partial x_i} = \frac{1}{n}\sum_{i=1}^{3}\frac{\partial P_{ik}^c}{\partial x_i} \approx n^{-1/3}\frac{\partial n}{\partial x_k}.$$

As consequence, as in the 1D case we find

$$U^c := \int_{-\infty}^{x_k}\left(\sum_{i=1}^{3}\frac{\partial U_{ik}^c}{\partial x_i}\right)dx_k \approx \int_{-\infty}^{x_k} n^{-1/3}\frac{\partial n}{\partial x_k}dx_k = |\Psi|^{4/3}$$

independently of $k = 1,2,3$.

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